Exercise 29

(a) If $F(x) = 5x/(1+x^2)$, find $F'(2)$ and use it to find an equation of the tangent line to the curve $y = 5x/(1+x^2)$ at the point $(2, 2)$.

(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

Solution

Determine the derivative of $F(x)$.

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{5(x+h)}{1+(x+h)^2} - \frac{5x}{1+x^2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{5(x+h)(1+x^2)}{[1+(x+h)^2](1+x^2)} - \frac{5x[1+(x+h)^2]}{[1+(x+h)^2](1+x^2)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{5(x+h)(1+x^2)-5x[1+(x+h)^2]}{[1+(x+h)^2](1+x^2)}}{h} \\
&= \lim_{h\to 0} \frac{5(x+h)(1+x^2) - 5x[1+(x+h)^2]}{h[1+(x+h)^2](1+x^2)} \\
&= \lim_{h\to 0} \frac{5(x+x^3+h+hx^2) - 5x[1+(x^2+2xh+h^2)]}{h[1+(x+h)^2](1+x^2)} \\
&= \lim_{h\to 0} \frac{(5x+5x^3+5h+5hx^2) - (5x+5x^3+10x^2h+5xh^2)}{h[1+(x+h)^2](1+x^2)} \\
&= \lim_{h\to 0} \frac{5h+5hx^2-10x^2h-5xh^2}{h[1+(x+h)^2](1+x^2)} \\
&= \lim_{h\to 0} \frac{5+5x^2-10x^2-5xh}{[1+(x+h)^2](1+x^2)} \\
&= \frac{5-5x^2}{(1+x^2)(1+x^2)} \\
&= \frac{5(1-x^2)}{(1+x^2)^2}
\end{aligned}$$

Plug in $x = 2$ to this formula to get $F'(2)$.

$$F'(2) = \frac{5[1-(2)^2]}{[1+(2)^2]^2} = -\frac{3}{5}$$

This is the slope of the tangent line to the curve at $x = 2$. Use the point-slope formula and the provided point $(2,2)$ to get the equation of this line.

$$y - 2 = -\frac{3}{5}(x-2)$$

$$y - 2 = -\frac{3}{5}x + \frac{6}{5}$$

$$y = -\frac{3}{5}x + \frac{16}{5}$$

Below is a graph of the curve along with the tangent line at $x = 2$.